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SCIENCE DIRECT
Journal of Sound and Vibration 274 (2004) 843–861
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

# Non-linear stochastic optimal control for coupled-structures system of multi-degree-of-freedom

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Received 13 November 2002; accepted 5 June 2003

**Abstract**

Coupled structures under random excitation are modelled as a quasi-integrable Hamiltonian system of multi-degree-of-freedom and the reduced-order model in structural mode space is formulated. A non-linear stochastic optimal control method for the system is presented. The non-linear optimal control of adjacent tall building structures coupled with supplemental control devices and under random seismic excitation is performed by using the proposed method. First, applying the stochastic averaging method to the system yields Itô stochastic differential equations for modal vibration energy processes, so that the system energy control is conducted generally instead of the system state control and the dimension of the control problem is reduced. Then applying the stochastic dynamical programming principle to the controlled diffusion processes yields a dynamical programming equation, taking into account random excitation spectra. An explicit polynomial solution to the equation is proposed to determine the non-linear optimal control forces. Furthermore, the response statistics of the controlled non-linear coupled structures under random seismic excitation are evaluated by using the stochastic averaging method, and are compared with those of the uncontrolled structures to determine the control efficacy. Numerical results illustrate the high control effectiveness and efficiency of the proposed non-linear stochastic optimal control method for coupled structures as a quasi-integrable Hamiltonian system.


## 1. Introduction

Active control of structural vibration induced by severe dynamic loading such as strong wind or earthquake ground motion has been an active research subject recently. A comprehensive survey on structural control research and application was given by Housner et al. [1]. The control

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doi:10.1016/S0022-460X(03)00610-2

effectiveness of structural systems is highly dependent on the used control method and a number of control methods were presented for designing active structural control law. In general, the optimal control method for structural systems based on the dynamical programming principle is more reasonable and effective than the others. Then the linear quadratic control method is frequently used in structural control, with a classical explicit solution of control law to the dynamical programming equation. However, the control method using non-linear optimal control law [2–6] has higher efficacy than that using linear one for structural response reduction, especially for strong-vibration response reduction. Since the dynamic loading such as wind or earthquake acting on engineering structures is random in nature, the optimal control method based on the stochastic dynamical programming principle [4–6] is more reasonable for structural control application. The recently proposed stochastic optimal control method has been applied to several simple systems with one- or two-degree-of-freedom under random excitation, and numerical results show that it is more effective and efficient than the others in random vibration control.

In dynamic response and control study, many engineering structures are usually modelled as integrable Hamiltonian systems. For example, the horizontal motion of a tall building structure [7] can be formulated as an integrable Hamiltonian system of multi-degree-of-freedom, especially in structural mode space. When structural or external damping and random loading such as wind or earthquake ground motion are considered, they become randomly excited and dissipated Hamiltonian systems called quasi-integrable Hamiltonian systems [8]. The Hamiltonian of structural systems represents total system energy generally while the independent integrals of motion indicate modal vibration energies and the structural energy control is an actual and effective approach to response reduction. Therefore, the non-linear stochastic optimal control of quasi-integrable Hamiltonian systems is a significant research subject.

On the other hand, in structural engineering field, interconnecting adjacent tall building structures with supplemental control devices is a practical and effective approach to mitigating structural seismic or wind response. The passive control of coupled adjacent structures with linear or non-linear devices has been widely studied [9–12], and the active or semi-active control of coupled adjacent structures under random seismic or wind excitation has been evolved [13–15]. Most of those studies used the linear quadratic control method for designing a control law of coupled structures, and consequently applying the non-linear stochastic optimal control method for quasi-integrable Hamiltonian systems to coupled structures under random seismic excitation is more interesting and would be more effective for seismic response mitigation.

The present study is focused firstly on the non-linear stochastic optimal control method for a quasi-integrable Hamiltonian system of multi-degree-of-freedom. The stochastic optimal control strategy proposed in Refs. [4–6] is applied to the system. The Itô stochastic differential equations for independent integrals of motion of the system as controlled diffusion processes are derived by using the stochastic averaging method. The dynamical programming equation for the controlled processes is established based on the stochastic dynamical programming principle, from which the non-linear optimal control law is determined in correspondence with a certain performance index. Then the developed control method is applied to coupled adjacent building structures for seismic response mitigation. A multi-degree-of-freedom model of the coupled structures with an arbitrary number of stories and with connecting control devices at any floors is formulated and converted into another one by using the modal transformation technique, in which the seismic excitation is modelled as a non-white Gaussian random process with the Kanai–Tajimi power spectrum. The non-linear optimal control forces for the coupled structures are obtained based on the stochastic averaging method and the stochastic dynamical programming principle. An explicit polynomial solution to the dynamical programming equation is proposed for the value function and corresponding performance index. Finally, the random seismic response of the controlled coupled structures is predicted by using the stochastic averaging method and compared with that of the uncontrolled structures to evaluate the control efficacy which is illustrated by the numerical results.

## 2. Quasi-integrable Hamiltonian systems and optimal control

Consider a controlled, randomly excited and dissipated Hamiltonian system of multi-degree-of-freedom, which is governed by $n$ pairs of equations of motion as follows:

$$\dot{Q}_i = \frac{\partial H}{\partial P_i}, \tag{1a}$$

$$\dot{P}_i = -\frac{\partial H}{\partial Q_i} - c_{ij}\frac{\partial H}{\partial P_j} + f_{ik}\xi_k(t) + u_i, \quad i,j = 1,2,\ldots,n;\ k = 1,2,\ldots,m, \tag{1b}$$

where $H = H(Q_i, P_i)$ is the Hamiltonian generally representing total energy of the system, $Q_i$ and $P_i$ are generalized displacement and momentum, respectively, $c_{ij} = c_{ij}(Q_i, P_i)$ denotes damping coefficient, $f_{ik}$ is the amplitude of random excitations and $\xi_k(t)$ is random process with zero mean and correlation functions $R_{kl}(\tau)$, $u_i$ represents non-linear control force to be determined by the stochastic dynamical programming principle.

The integrability of a Hamiltonian system depends on the structure of its Hamiltonian [8,16]. It is assumed herein that the Hamiltonian system corresponding to system (1) is completely integrable, as many engineering structures are modelled generally. Then there exist $n$ independent integrals of motion $H_i$ $(i = 1,2,\ldots,n)$ which are in involution and the energy distribution among various degrees of freedom as well as the total system energy is adjustable. For the quasi-integrable Hamiltonian system, its stationary probability density is a functional of independent integrals of motion and thus, the total energy and energy distribution can be controlled by non-linear control forces as well as changed by dampings and excitations. The system vibration can be mitigated by the system energy control rather than the system state control.

To conduct the system energy control based on the stochastic dynamical programming principle, the stochastic averaging method for quasi-integrable Hamiltonian systems [8,17] is first applied to system (1) to yield diffusion processes generally indicating modal vibration energies. In the case that the corresponding Hamiltonian system is non-resonant, the averaged Itô stochastic differential equations for independent integrals of motion $H_i$ $(i = 1,2,\ldots,n)$ as an $n$-dimensional vector diffusion process are of the form

$$\mathrm{d}H_r = \left[m_r(\mathbf{H}) + \left\langle \frac{\partial H_r}{\partial P_i} u_i \right\rangle_t\right]\mathrm{d}t + \sigma_{rk}(\mathbf{H})\,\mathrm{d}B_k(t), \quad r,i =,1,2,\ldots,n;\ k = 1,2,\ldots,m, \tag{2}$$

where $\mathbf{H} = [H_1, H_2, \ldots, H_n]^{\mathrm{T}}$, $\langle \cdot \rangle_t$ denotes the time-averaging operator, $B_k(t)$ $(k = 1, 2, \ldots, m)$ are independent unit Wiener processes; the drift coefficients and diffusion coefficients are

$$m_r(\mathbf{H}) = \Bigg\langle - c_{ij}\frac{\partial H_r}{\partial P_i}\frac{\partial H_r}{\partial P_j} + \int_{-\infty}^{0}\Bigg[\left(\frac{\partial H_s}{\partial P_j}f_{jl}\right)_{t+\tau}\frac{\partial}{\partial H_s}\left(\frac{\partial H_r}{\partial P_i}f_{ik}\right)_t + \left(\frac{\partial \theta_s}{\partial P_j}f_{jl}\right)_{t+\tau}\frac{\partial}{\partial \theta_s}\left(\frac{\partial H_r}{\partial P_i}f_{ik}\right)_t\Bigg]R_{kl}(\tau)\,\mathrm{d}\tau\Bigg\rangle_t \tag{3a}$$

$$\sigma_{ru}(\mathbf{H})\sigma_{su}(\mathbf{H}) = \left\langle \int_{-\infty}^{\infty}\left(\frac{\partial H_s}{\partial P_j}f_{jl}\right)_{t+\tau}\left(\frac{\partial H_r}{\partial P_i}f_{ik}\right)_t R_{kl}(\tau)\,\mathrm{d}\tau\right\rangle_t$$
$$r, s, i, j = 1, 2, \ldots, n; \quad k, l, u = 1, 2, \ldots, m \tag{3b}$$

in which $\theta_s$ is generalized phase process. The time averaging can be replaced by the phase-space averaging with respect to $q_i$. For Gaussian white noise processes $\xi_k(t)$ $(k = 1, 2, \ldots, m)$ with intensities $2D_{kl}$, the drift coefficients and diffusion coefficients become

$$m_r(\mathbf{H}) = \frac{1}{T(\mathbf{H})}\oint\left[\left(-c_{ij}\frac{\partial H_r}{\partial p_i}\frac{\partial H_r}{\partial p_j} + D_{kl}f_{ik}f_{jl}\frac{\partial^2 H_r}{\partial p_i \partial p_j}\right)\Big/\left(\frac{\partial H_1}{\partial p_1}\frac{\partial H_2}{\partial p_2}\cdots\frac{\partial H_n}{\partial p_n}\right)\right]\mathrm{d}q_1\,\mathrm{d}q_2\cdots\mathrm{d}q_n, \tag{4a}$$

$$\sigma_{ru}(\mathbf{H})\sigma_{su}(\mathbf{H}) = \frac{1}{T(\mathbf{H})}\oint\left[\left(2D_{kl}f_{ik}f_{jl}\frac{\partial H_r}{\partial p_i}\frac{\partial H_s}{\partial p_j}\right)\Big/\left(\frac{\partial H_1}{\partial p_1}\frac{\partial H_2}{\partial p_2}\cdots\frac{\partial H_n}{\partial p_n}\right)\right]\mathrm{d}q_1\,\mathrm{d}q_2\cdots\mathrm{d}q_n, \tag{4b}$$

$$T(\mathbf{H}) = \oint\left[1\Big/\left(\frac{\partial H_1}{\partial p_1}\frac{\partial H_2}{\partial p_2}\cdots\frac{\partial H_n}{\partial p_n}\right)\right]\mathrm{d}q_1\,\mathrm{d}q_2\cdots\mathrm{d}q_n. \tag{4c}$$

The random response control of system (1) can be achieved by the energy control of corresponding diffusion processes (2) and the dimension of the control problem is reduced from Eqs. (1) to (2). The optimal control of diffusion processes (2) can be performed based on the stochastic dynamical programming principle [18–20]. The optimal control law depends on the objective of system control, which is expressed in terms of performance index. For **H** control, the performance index in finite time interval is

$$J = E\left[\int_0^{t_f} L(\mathbf{H}(\tau), \langle g_1(\mathbf{u}(\tau))\rangle_t)\,\mathrm{d}\tau + \Psi(\mathbf{H}(t_f))\right], \tag{5}$$

where $E[\cdot]$ denotes the expectation operator, $t_f$ is terminal time, $L(\mathbf{H}, \langle g_1 \rangle_t)$ and $g_1(\mathbf{u})$ represent continuous differential convex functions, $\mathbf{u} = [u_1, u_2, \ldots, u_n]^{\mathrm{T}}$ and $\Psi(t_f)$ represents a terminal cost function. In infinite time-interval ergodic control, the performance index (5) becomes

$$J = \lim_{t_f \to \infty}\frac{1}{t_f}\int_0^{t_f} L(\mathbf{H}(\tau), \langle g_1(\mathbf{u}(\tau))\rangle_t)\,\mathrm{d}\tau. \tag{6}$$

Obviously, the performance index $J$ depends on the used function $L$ of **H** and **u**. For a linear system, $H_r$ is quadratic in both generalized displacement and momentum. Then $L$ would be a function of quadratic system states, and linear $L$ is similar to that used in the conventional LQG control [18]. For a convex function $L$, the diffusion process vector **H** in entire dynamic process decreases in correspondence with function $L$ and performance index $J$. Thus the random response can be reduced by minimizing the performance index. Applying the stochastic dynamical programming principle [18–20] yields a dynamical programming equation, for example, to the controlled processes (2) with performance index (5) as follows:

$$\frac{\partial V}{\partial t} = -\min_{\mathbf{u}}\left\{ L(\mathbf{H}, \langle g_1(\mathbf{u})\rangle_t) + \left[ m_r(\mathbf{H}) + \left\langle u_i \frac{\partial H_r}{\partial P_i} \right\rangle_t \right] \frac{\partial V}{\partial H_r} + \tfrac{1}{2}\sigma_{rk}(\mathbf{H})\sigma_{sk}(\mathbf{H}) \frac{\partial^2 V}{\partial H_r \partial H_s} \right\} \quad (7)$$

or to the controlled processes (2) with performance index (6) as

$$\lambda = \min_{\mathbf{u}}\left\{ L(\mathbf{H}, \langle g_1(\mathbf{u})\rangle_t) + \left[ m_r(\mathbf{H}) + \left\langle u_i \frac{\partial H_r}{\partial P_i} \right\rangle_t \right] \frac{\partial V}{\partial H_r} + \tfrac{1}{2}\sigma_{rk}(\mathbf{H})\sigma_{sk}(\mathbf{H}) \frac{\partial^2 V}{\partial H_r \partial H_s} \right\}, \quad (8)$$

where $V = V(\mathbf{H}, t)$ is called value function and $\lambda$ is constant. The random excitation spectra are taken into account by the drift coefficients and diffusion coefficients (3) in the control process of Eq. (7) or (8).

The optimal control law can be determined by minimizing the right-hand side of Eq. (7) or (8). Its governing equations are

$$\frac{\partial}{\partial u_i}\left[ L(\mathbf{H}, \langle g_1(\mathbf{u})\rangle_t) + \left\langle u_i \frac{\partial H_r}{\partial P_i} \right\rangle_t \frac{\partial V}{\partial H_r} \right] = 0, \quad i = 1, 2, \ldots, n. \quad (9)$$

In general, let function $L$ be quadratic in control force vector $\mathbf{u}$, that is

$$L(\mathbf{H}, \langle g_1(\mathbf{u})\rangle_t) = g(\mathbf{H}) + \langle \mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u}\rangle_t, \quad (10)$$

where $g(\mathbf{H}) \geqslant 0$ and $\mathbf{R}$ is a positive-definite symmetric matrix. Then the optimal control forces are obtained as follows:

$$u_i^* = -\tfrac{1}{2} R_{ij}^{-1} \frac{\partial H_r}{\partial P_j} \frac{\partial V}{\partial H_r}, \quad i = 1, 2, \ldots, n \quad (11)$$

which depend on value function $V$. Since $V$ is generally a non-linear functional of $H_r$ and $H_r$ is a non-linear function of $Q_i$ and $P_i$, the obtained optimal control forces (11) would be non-linear in generalized displacements and momenta and thus, determine a non-linear stochastic optimal control of the quasi-integrable Hamiltonian system. In the case that the Hamiltonian is the sum of $n$ independent integrals of motion, that is, $H = H_1(Q_1, P_1) + H_2(Q_2, P_2) + \cdots + H_n(Q_n, P_n)$, the optimal control forces (11) become

$$u_i^* = -\tfrac{1}{2} R_{ir}^{-1} \frac{\partial V}{\partial H_r} \dot{Q}_r, \quad i = 1, 2, \ldots, n. \quad (12)$$

Eq. (12) implies that optimal control force $u_i^*$ is a quasi-linear function of generalized velocity and may be dissipative.

By substituting the expression of $u_i^*$ obtained from Eq. (9) or (11) into Eqs. (7) and (8), the dynamical programming equations become for the value function, respectively as

$$\frac{\partial V}{\partial t} + L(\mathbf{H}, \langle g_1(\mathbf{u}^*)\rangle_t) + \left[ m_r(\mathbf{H}) + \left\langle u_i^* \frac{\partial H_r}{\partial P_i} \right\rangle_t \right] \frac{\partial V}{\partial H_r} + \tfrac{1}{2}\sigma_{rk}(\mathbf{H})\sigma_{sk}(\mathbf{H}) \frac{\partial^2 V}{\partial H_r \partial H_s} = 0, \quad (13)$$

$$L(\mathbf{H}, \langle g_1(\mathbf{u}^*)\rangle_t) + \left[ m_r(\mathbf{H}) + \left\langle u_i^* \frac{\partial H_r}{\partial P_i} \right\rangle_t \right] \frac{\partial V}{\partial H_r} + \tfrac{1}{2}\sigma_{rk}(\mathbf{H})\,\sigma_{sk}(\mathbf{H}) \frac{\partial^2 V}{\partial H_r \partial H_s} = \lambda. \quad (14)$$

Here $V$ as a function of $H_r$ can be obtained by solving Eq. (13) for the finite time-interval control or (14) for the infinite time-interval control, and then the non-linear optimal control force $u_i^*$ can be determined. Due to the dynamical programming Eq. (7) or (8) with a classical solution [19], the uniform expression of optimal control forces can be derived and the value function would be smooth continuous and may be calculated from Eq. (13) or (14) by using a conventional numerical technique.

## 3. Coupled building structures under random seismic excitation

To illustrate the application and effectiveness of the proposed non-linear stochastic optimal control method for quasi-integrable Hamiltonian systems of multi-degree-of-freedom, consider two adjacent high-rise building structures with $n_1$ and $n_2$ ($n_1 \geqslant n_2$) stories and interconnected by control devices at $n_3$ ($n_3 \leqslant n_2$) floors as shown in Fig. 1. It is assumed that the coupled structures are subjected to a lateral seismic excitation and the control forces are also in horizontal direction. In the case of linear elastic shear-type structures, the equations of motion of the coupled structural system are represented by

$$M_1\ddot{X}_1 + C_1\dot{X}_1 + K_1X_1 = -\ddot{x}_g(t)M_1E_1 + P_1U, \tag{15a}$$

$$M_2\ddot{X}_2 + C_2\dot{X}_2 + K_2X_2 = -\ddot{x}_g(t)M_2E_2 + P_2U, \tag{15b}$$

where $X_i$ $(i = 1, 2)$ denote the $n_i$-dimensional lateral displacement vectors, $M_i$, $C_i$ and $K_i$ $(i = 1, 2)$ are the $n_i \times n_i$-dimensional symmetric positive-definite mass, damping and stiffness matrices of structure $i$, respectively, $E_i$ $(i = 1, 2)$ are the $n_i$-dimensional vectors with unit elements, $\ddot{x}_g(t)$ denotes the random ground acceleration excitation modelled as a non-white Gaussian process with the Kanai–Tajimi power spectrum [21,22], $U$ represents the $n_3$-dimensional coupling control force vector and $P_i$ $(i = 1, 2)$ are the $n_i \times n_3$-dimensional matrices indicating the installation position of control devices. Note that the control forces of the coupled structures are exerted upon each other inversely with the relation $P_1 = [0, -P_2^{\mathrm{T}}]^{\mathrm{T}}$. For convenience, matrix and vector symbols are not in boldface here and hereafter.

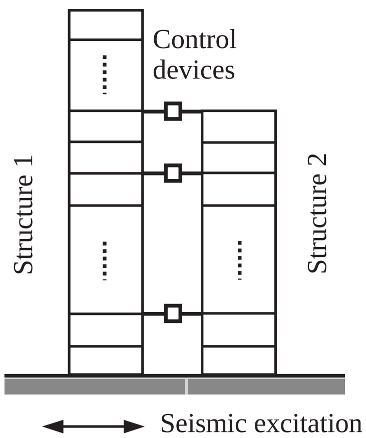



Fig. 1. Coupled building structures.

The random response of the coupled structures can be expressed using the concept of sub-structuring and the modes of the corresponding uncoupled structures in the assumption that the effect of higher-order modes on response is so slight as to be neglected. For the response control, the first $m_{i+2}$ ($m_{i+2} \leqslant n_i$, $i = 1, 2$) dominant modes of structure $i$ are aimed at and taken to assemble into a reduced mode matrix $\bar{\Phi}_i$ normalized with respect to the mass matrix $M_i$. By using the modal transformation technique, Eqs. (15a) and (15b) of the coupled structures are converted into

$$\ddot{q}_{1i} + 2\zeta_{1i}\omega_{1i}\dot{q}_{1i} + \omega_{1i}^2 q_{1i} = -\beta_{1i}\ddot{x}_g(t) + v_{1i}, \quad i = 1, 2, \ldots, m_3, \tag{16a}$$

$$\ddot{q}_{2i} + 2\zeta_{2i}\omega_{2i}\dot{q}_{2i} + \omega_{2i}^2 q_{2i} = -\beta_{2i}\ddot{x}_g(t) + v_{2i}, \quad i = 1, 2, \ldots, m_4, \tag{16b}$$

where $q_{ji}$ denotes the $i$th element of the modal displacement vector of structure $j$ $(j = 1, 2)$, $\omega_{ji}$ and $\zeta_{ji}$ $(j = 1, 2)$ are the $i$th modal frequencies and damping ratios, respectively, $\beta_{ji} = \phi_{ji}^{\mathrm{T}} M_j E_j$ $(j = 1, 2)$ are the coefficients of the $i$th modal excitation and $v_{ji} = \phi_{ji}^{\mathrm{T}} P_j U$ $(j = 1, 2)$ denote the control forces corresponding to the $i$th mode, in which $\phi_{ji}$ is the $i$th mode vector in structural mode matrix $\bar{\Phi}_j$ $(j = 1, 2)$.

The structural system corresponding to Eq. (16a) or (16b) without damping, excitation and control forces is linear and can be modelled as an integrable Hamiltonian system which is non-resonant generally. The Hamiltonian $\tilde{H}_j$ is equal to the sum of independent integrals of motion $H_{ji}$, that is

$$\tilde{H}_j = \sum_{i=1}^{m_{i+2}} H_{ji}, \quad j = 1, 2, \tag{17a}$$

$$H_{ji} = (\dot{q}_{ji}^2 + \omega_{ji}^2 q_{ji}^2)/2, \tag{17b}$$

where $\tilde{H}_j$ and $H_{ji}$ represent total vibration energy and modal vibration energy of structure $j$, respectively. Then the coupled structural system (16) is expressed as a controlled, randomly excited and dissipated Hamiltonian system of multi-degree-of-freedom with the separable Hamiltonian (17). The modal vibration energies as well as total energy are adjustable and can be controlled by the non-linear control forces.

Applying the stochastic averaging method for quasi-integrable Hamiltonian systems [8,17] to (16) yields the Itô stochastic differential equation for the modal vibration energies as follows:

$$\mathrm{d}\bar{H} = \left[\bar{m}(\bar{H}) + \left\langle \frac{\partial \bar{H}}{\partial \dot{\bar{Q}}} \bar{U}_v \right\rangle_t \right] \mathrm{d}t + \bar{\sigma}(\bar{H})\, \mathrm{d}\bar{W}(t), \tag{18}$$

where the modal vibration energy vector $\bar{H}$ of the coupled structural system, modal displacement vector $\bar{Q}$, modal control force vector $\bar{U}_v$, drift coefficient vector $\bar{m}(\bar{H})$, diffusion coefficient matrix $\bar{\sigma}(\bar{H})$ and unit Wiener process vector $\bar{W}(t)$ are respectively

$$\bar{H} = [\bar{H}_1^{\mathrm{T}}, \bar{H}_2^{\mathrm{T}}]^{\mathrm{T}} = [H_{11}, H_{12}, \ldots, H_{1m_3}, H_{21}, H_{22}, \ldots, H_{2m_4}]^{\mathrm{T}}, \tag{19a}$$

$$\bar{Q} = [\bar{Q}_1^{\mathrm{T}}, \bar{Q}_2^{\mathrm{T}}]^{\mathrm{T}} = [q_{11}, q_{12}, \ldots, q_{1m_3}, q_{21}, q_{22}, \ldots, q_{2m_4}]^{\mathrm{T}}, \tag{19b}$$

$$\bar{U}_v = [v_{11}, v_{12}, \ldots, v_{1m_3}, v_{21}, v_{22}, \ldots, v_{2m_4}]^{\mathrm{T}} = [P_1^{\mathrm{T}}\bar{\Phi}_1, P_2^{\mathrm{T}}\bar{\Phi}_2]^{\mathrm{T}} U, \tag{19c}$$

$$\bar{m}(\bar{H}) = [\bar{m}_1^{\mathrm{T}}, \bar{m}_2^{\mathrm{T}}]^{\mathrm{T}} = [m_{11}, m_{12}, \ldots, m_{1m_3}, m_{21}, m_{22}, \ldots, m_{2m_4}]^{\mathrm{T}}, \tag{19d}$$

$$\bar{\sigma}(\bar{H}) = \text{diag}\{\bar{\sigma}_1, \bar{\sigma}_2\} = \text{diag}\{\sigma_{11}, \sigma_{12}, \ldots, \sigma_{1m_3}, \sigma_{21}, \sigma_{22}, \ldots, \sigma_{2m_4}\}, \tag{19e}$$

$$\bar{W}(t) = [W_{11}, W_{12}, \ldots, W_{1m_3}, W_{21}, W_{22}, \ldots, W_{2m_4}]^{\mathrm{T}} \tag{19f}$$

with

$$m_{1i}(H_{1i}) = -2\zeta_{1i}\omega_{1i}H_{1i} + \tfrac{1}{2}\beta_{1i}^2 S_g(\omega_{1i}), \quad m_{2i}(H_{2i}) = -2\zeta_{2i}\omega_{2i}H_{2i} + \tfrac{1}{2}\beta_{2i}^2 S_g(\omega_{2i}), \tag{20a}$$

$$\sigma_{1i}^2(H_{1i}) = \beta_{1i}^2 H_{1i} S_g(\omega_{1i}), \quad \sigma_{2i}^2(H_{2i}) = \beta_{2i}^2 H_{2i} S_g(\omega_{2i}), \tag{20b}$$

$$S_g(\omega) = \sigma^2 \frac{\omega_g^4 + 4\zeta_g^2\omega_g^2\omega^2}{(\omega_g^2 - \omega^2)^2 + 4\zeta_g^2\omega_g^2\omega^2}. \tag{20c}$$

Here $S_g(\omega)$ is the Kanai–Tajimi power spectral density of random seismic excitation. The Itô equation (18) implies that the averaged modal vibration energy $\bar{H}$ is a controlled vector diffusion process. The random response control of the coupled structural system (15) can be achieved by the energy control of the corresponding diffusion processes (18) and therefore, the dimension of the control problem is reduced from $2(n_1 + n_2)$ to $m_3 + m_4$.

## 4. Non-linear optimal control law and response prediction

In this study, it is assumed that the system states such as displacements and velocities or modal vibration energies associated with (18) can be determined exactly by measurement. Then the optimal control problem is independent of the state observation problem and the optimal control law can be determined directly based on the stochastic dynamical programming principle [18–20]. For the system energy ($\bar{H}$) control, performance indexes can be expressed as Eqs. (5) and (6). The performance index in infinite time interval is of the form

$$J = \lim_{t_f \to \infty} \frac{1}{t_f} \int_0^{t_f} L[\bar{H}(\tau), \langle g_1(U(\tau)) \rangle_t] \, \mathrm{d}\tau. \tag{21}$$

Applying the stochastic dynamical programming principle to the controlled energy processes (18) with performance index (21) yields the following dynamical programming equation

$$\lambda = \min_U \left\{ L[\bar{H}, \langle g_1(U) \rangle_t] + \left(\frac{\partial V}{\partial \bar{H}}\right)^{\mathrm{T}} \left[ \bar{m}(\bar{H}) + \left\langle \frac{\partial \bar{H}}{\partial \dot{\bar{Q}}} \bar{U}_v \right\rangle_t \right] + \frac{1}{2} tr \left[ \frac{\partial^2 V}{\partial \bar{H}^2} \bar{\sigma}(\bar{H}) \bar{\sigma}^{\mathrm{T}}(\bar{H}) \right] \right\}, \tag{22}$$

where $tr[\cdot]$ denotes the trace operator of square matrix. The modal control force vector $\bar{U}_v$ is represented in terms of the coupled structural control force vector $U$ by Eq. (19c). It is seen from Eqs. (22), (19d), (19e) and (20) that the random seismic excitation spectrum is incorporated in the control process.

With the similarity to Eqs. (9)–(14), by minimizing the right-hand side of Eq. (22), the equation for optimal control law is obtained as follows:

$$\frac{\partial}{\partial U} \left\{ L[\bar{H}, \langle g_1(U) \rangle_t] + \left\langle U^{\mathrm{T}} [P_1^{\mathrm{T}} \bar{\Phi}_1, P_2^{\mathrm{T}} \bar{\Phi}_2] \frac{\partial \bar{H}}{\partial \dot{\bar{Q}}} \right\rangle_t \frac{\partial V}{\partial \bar{H}} \right\} = 0. \tag{23}$$

For function

$$L = g(\bar{H}) + \langle \bar{U}_v^{\mathrm{T}} R \bar{U}_v \rangle_t = g(\bar{H}_1, \bar{H}_2) + \langle U^{\mathrm{T}} R_p U \rangle_t,$$

in which $R_p = [P_1^{\mathrm{T}}\bar{\Phi}_1, P_2^{\mathrm{T}}\bar{\Phi}_2]R[P_1^{\mathrm{T}}\bar{\Phi}_1, P_2^{\mathrm{T}}\bar{\Phi}_2]^{\mathrm{T}}$, the optimal control force vector is

$$U^* = -\frac{1}{2}R_p^{-1}\left(P_1^{\mathrm{T}}\bar{\Phi}_1 \frac{\partial \bar{H}_1}{\partial \dot{\bar{Q}}_1}\frac{\partial V}{\partial \bar{H}_1} + P_2^{\mathrm{T}}\bar{\Phi}_2 \frac{\partial \bar{H}_2}{\partial \dot{\bar{Q}}_2}\frac{\partial V}{\partial \bar{H}_2}\right). \tag{24}$$

It is always possible to select control devices and a reduced structural system such that the dimension of the reduced system is not less than the number of the control devices, that is, $m_3 + m_4 \geqslant n_3$ and the rank of matrix $[P_1^{\mathrm{T}}\bar{\Phi}_1, P_2^{\mathrm{T}}\bar{\Phi}_2]_{n_3\times(m_3+m_4)}$ is equal to $n_3$ for a certain placement of the control devices. In this case, matrix $R_p$ would be symmetric positive definite. The optimal control forces (24) act as generalized non-linear damping forces since the value function $V$ is generally a non-linear functional of the modal vibration energies $H_{ji}$ $(j = 1, 2)$ and the partial derivative of the $i$th mode vibration energy $H_{ji}$ with respect to modal velocity $\dot{q}_{ji}$ is just the corresponding modal velocity $\dot{q}_{ji}$.

By substituting the optimal control forces (24) into the dynamical programming equation (22) and averaging terms involving the control forces, the following equation for the value function is obtained:

$$\begin{aligned}\lambda = g(\bar{H}_1, \bar{H}_2) &+ \left(\frac{\partial V}{\partial \bar{H}_1}\right)^{\mathrm{T}}\left[\bar{m}_1(\bar{H}_1) + \tfrac{1}{2}\bar{m}_{u1}(\bar{H})\right] + \left(\frac{\partial V}{\partial \bar{H}_2}\right)^{\mathrm{T}}\left[\bar{m}_2(\bar{H}_2) + \tfrac{1}{2}\bar{m}_{u2}(\bar{H})\right] \\ &+ \tfrac{1}{2}\,tr\left[\frac{\partial^2 V}{\partial \bar{H}_1^2}\bar{\sigma}_1(\bar{H}_1)\bar{\sigma}_1^{\mathrm{T}}(\bar{H}_1)\right] + \tfrac{1}{2}\,tr\left[\frac{\partial^2 V}{\partial \bar{H}_2^2}\bar{\sigma}_2(\bar{H}_2)\bar{\sigma}_2^{\mathrm{T}}(\bar{H}_2)\right],\end{aligned} \tag{25}$$

where

$$\bar{m}_{u1}(\bar{H}) = -\frac{1}{2}\left\{\begin{matrix} \phi_{11}^{\mathrm{T}}P_u\phi_{11}H_{11}\dfrac{\partial V}{\partial H_{11}} \\ \phi_{12}^{\mathrm{T}}P_u\phi_{12}H_{12}\dfrac{\partial V}{\partial H_{12}} \\ \vdots \\ \phi_{1m_3}^{\mathrm{T}}P_u\phi_{1m_3}H_{1m_3}\dfrac{\partial V}{\partial H_{1m_3}} \end{matrix}\right\}, \quad \bar{m}_{u2}(\bar{H}) = -\frac{1}{2}\left\{\begin{matrix} \phi_{21}^{\mathrm{T}}P_w\phi_{21}H_{21}\dfrac{\partial V}{\partial H_{21}} \\ \phi_{22}^{\mathrm{T}}P_w\phi_{22}H_{22}\dfrac{\partial V}{\partial H_{22}} \\ \vdots \\ \phi_{2m_4}^{\mathrm{T}}P_w\phi_{2m_4}H_{2m_4}\dfrac{\partial V}{\partial H_{2m_4}} \end{matrix}\right\}, \tag{26a}$$

$$P_u = P_1 R_p^{-1} P_1^{\mathrm{T}}, \quad P_w = P_2 R_p^{-1} P_2^{\mathrm{T}}. \tag{26b}$$

The value function $V$ can be obtained from solving Eq. (25) with a classical solution and the non-linear optimal control force $U^*$ is then determined as a function of modal vibration energy $\bar{H}$ or modal displacement $\bar{Q}$ and modal velocity $\dot{\bar{Q}}$. Actually, it is difficult to directly solve the partial differential equation (25) so that an alternative approach is adopted to convert the equation into algebraic equations for solving. Take function $g$ in the following polynomial form:

$$g(\bar{H}_1, \bar{H}_2) = s_0 + \sum_{i=1}^{m_3} s_{1i}^a H_{1i} + \sum_{i=1}^{m_4} s_{2i}^a H_{2i} + \sum_{i=1}^{m_3} s_{1i}^b H_{1i}^2 + \sum_{i=1}^{m_4} s_{2i}^b H_{2i}^2 + \sum_{i=1}^{m_3} s_{1i}^c H_{1i}^3 + \sum_{i=1}^{m_4} s_{2i}^c H_{2i}^3 + \sum_{i \neq j}^{m_3} s_{1ij}^b H_{1i} H_{1j} + \sum_{i \neq j}^{m_4} s_{2ij}^b H_{2i} H_{2j} + \sum_{i,j=1}^{m_3,m_4} s_{3ij}^b H_{1i} H_{2j} + O(H_{i_1 j_1} H_{i_2 j_2} H_{i_3 j_3}), \tag{27}$$

where $s_{1ij}^b = s_{1ji}^b$, $s_{2ij}^b = s_{2ji}^b$ and the weight coefficients are non-negative. Then the corresponding value function solution $V$ in polynomial form can be determined by

$$V(\bar{H}_1, \bar{H}_2) = \sum_{i=1}^{m_3} p_{1i}^a H_{1i} + \sum_{i=1}^{m_4} p_{2i}^a H_{2i} + \sum_{i=1}^{m_3} p_{1i}^b H_{1i}^2 + \sum_{i=1}^{m_4} p_{2i}^b H_{2i}^2 + \sum_{i \neq j}^{m_3} p_{1ij}^b H_{1i} H_{1j} + \sum_{i \neq j}^{m_4} p_{2ij}^b H_{2i} H_{2j} + \sum_{i,j=1}^{m_3,m_4} p_{3ij}^b H_{1i} H_{2j}, \tag{28}$$

where $p_{1ij}^b = p_{1ji}^b$ and $p_{2ij}^b = p_{2ji}^b$. Substituting Eqs. (27) and (28) into Eq. (25) and comparing the coefficients in terms of the power of modal vibration energies yield a series of algebraic equations. For certain $s_{1i}^a$, $s_{2i}^a$, $s_{1i}^c$, $s_{2i}^c$, $s_{1ij}^b$, $s_{2ij}^b$, and $s_{3ij}^b$, the weight coefficients in value function $V$ can be obtained from solving those algebraic equations so that the value function (28) and the optimal control forces (24) are determined eventually.

To evaluate the control efficacy of the proposed non-linear stochastic optimal control method, the random response of the controlled structural system under seismic excitation is further predicted and compared with that of the uncontrolled structural system in terms of performance criteria. The controlled structural system is non-linear due to the non-linear optimal control forces and the stochastic averaging method for quasi-integrable Hamiltonian systems can be used. For the response analysis, the first $m_i$ ($m_{i+2} \leqslant m_i \leqslant n_i$, $i = 1, 2$) important modes of structure $i$ are taken and assembled into a reduced mode matrix $\Phi_i$ normalized with respect to mass matrix $M_i$. By substituting the optimal control forces (24) into the coupled structural equation (15), using the modal transformation technique to convert the equation into one in the modal space and then applying the stochastic averaging method to it, the Itô stochastic differential equations for the modal vibration energies are obtained as follows:

$$\mathrm{d}H = [m(H) + m_u(\bar{H})]\,\mathrm{d}t + \sigma(H)\,\mathrm{d}W(t), \tag{29}$$

where $H$ is the $(m_1 + m_2)$-dimensional modal vibration energy vector, $W(t)$ is the $(m_1 + m_2)$-dimensional unit Wiener process vector, $\sigma(H)$ is the $(m_1 + m_2) \times (m_1 + m_2)$-dimensional diagonal diffusion coefficient matrix, $m(H)$ and $m_u(\bar{H})$ are the $(m_1 + m_2)$-dimensional drift coefficient vectors independent of the control forces and involving the control forces, respectively. The drift coefficients in $m(H)$ and diffusion coefficients in $\sigma(H)$ are given by Eqs. (20a) and (20b). $m_u(\bar{H}) = [m_{u1}^{\mathrm{T}}, m_{u2}^{\mathrm{T}}]^{\mathrm{T}}$. The first $m_3$ elements of vector $m_{u1}$ and first $m_4$ elements of vector $m_{u2}$ are represented by Eq. (26a). The other elements of $m_{u1}$ and $m_{u2}$ are equal to zeros due to the averaged Itô equations of the linear uncontrolled structural system corresponding to Eq. (29) separable and the value function $V$ independent of modal vibration energies $H_{1i}$ $(i = m_3 + 1, \ldots, m_1)$ and $H_{2i}$ $(i = m_4 + 1, \ldots, m_2)$. Thus the optimal control forces affect only the controlled reduced modal energy processes in the sense of stochastic averaging.

The Fokker–Planck–Kolmogorov (FPK) equation associated with the Itô equation (29) can be established. The stationary FPK equation is

$$\sum_{i=1}^{m_1} \frac{\partial}{\partial H_{1i}}\left\{ \left[-m_{1i}(H_{1i}) + \tfrac{1}{2}\phi_{1i}^{\mathrm{T}} P_u \phi_{1i} H_{1i} \frac{\partial V}{\partial H_{1i}}\right] p + \frac{1}{2}\frac{\partial}{\partial H_{1i}}[\sigma_{1i}^2(H_{1i})p]\right\} + \sum_{i=1}^{m_2} \frac{\partial}{\partial H_{2i}}\left\{ \left[-m_{2i}(H_{2i}) + \tfrac{1}{2}\phi_{2i}^{\mathrm{T}} P_w \phi_{2i} H_{2i} \frac{\partial V}{\partial H_{2i}}\right] p + \frac{1}{2}\frac{\partial}{\partial H_{2i}}[\sigma_{2i}^2(H_{2i})p]\right\} = 0 \tag{30}$$

with a stationary probability density solution

$$p(H_1, H_2) = C_p \exp\{-\varphi(H_1, H_2)\}, \tag{31}$$

where $C_p$ is a normalization constant and the probability potential

$$\varphi(H_1, H_2) = \int_0^{(H_i, H_2)} \sum_{i=1}^{m_1} \frac{\partial \varphi}{\partial H_{1i}} \mathrm{d}H_{1i} + \sum_{i=1}^{m_2} \frac{\partial \varphi}{\partial H_{2i}} \mathrm{d}H_{2i}, \tag{32a}$$

$$\frac{\partial \varphi}{\partial H_{1i}} = \frac{\partial \sigma_{1i}^2/\partial H_{1i} - 2m_{1i} + \phi_{1i}^{\mathrm{T}} P_u \phi_{1i} H_{1i} \partial V/\partial H_{1i}}{\sigma_{1i}^2}, \tag{32b}$$

$$\frac{\partial \varphi}{\partial H_{2i}} = \frac{\partial \sigma_{2i}^2/\partial H_{2i} - 2m_{2i} + \phi_{2i}^{\mathrm{T}} P_w \phi_{2i} H_{2i} \partial V/\partial H_{2i}}{\sigma_{2i}^2}. \tag{32c}$$

The mean square (MS) modal displacement and modal velocity are obtained by using the probability density (31) as follows:

$$E[q_{jk}^2] = \frac{1}{\omega_{jk}^2} \int_0^{+\infty} H_{jk} p(H_1, H_2)\, \mathrm{d}H_1\, \mathrm{d}H_2, \tag{33a}$$

$$E[\dot{q}_{jk}^2] = \int_0^{+\infty} H_{jk} p(H_1, H_2)\, \mathrm{d}H_1\, \mathrm{d}H_2. \tag{33b}$$

Then the MS displacement, interstorey drift, base shear and optimal control force of the controlled coupled structures are represented based on the modal transformation technique by

$$E[x_{ji}^2] = \sum_{k=1}^{m_j} (\phi_{jk}^i)^2 E[q_{jk}^2], \tag{34a}$$

$$E[(x_{ji} - x_{j,i-1})^2] = \sum_{k=1}^{m_j} (\phi_{jk}^i - \phi_{jk}^{i-1})^2 E[q_{jk}^2], \tag{34b}$$

$$
\begin{aligned}
&E\left[\left(\sum_{i=1}^{n_j} m_{jii}(\ddot{x}_{ji}+\ddot{x}_g)+(-1)^{j+1}\sum_{i=1}^{n_3} u_i^*\right)^2\right] \\
&\quad = E[(E_j^{\mathrm{T}} M_j(\ddot{X}_j+\ddot{x}_g E_j)-E_j^{\mathrm{T}} P_j U^*)^2] \\
&\quad = \sum_{k=1}^{m_j}\left(\sum_{i=1}^{n_j} m_{jii}\phi_{jk}^i\right)^2 (\omega_{jk}^4 E[q_{jk}^2]+4\zeta_{jk}^2\omega_{jk}^2 E[\dot{q}_{jk}^2]),
\end{aligned}
\tag{34c}
$$

$$
\begin{aligned}
E[u_i^{*2}] = \tfrac{1}{4} R_{pi}^{-1}\Bigg[ & P_1^{\mathrm{T}}\left(\sum_{k=1}^{m_3} \phi_{1k}\phi_{1k}^{\mathrm{T}} E\left[\dot{q}_{1k}^2\left(\frac{\partial V}{\partial H_{1k}}\right)^2\right]\right)P_1 \\
& + P_2^{\mathrm{T}}\left(\sum_{k=1}^{m_4} \phi_{2k}\phi_{2k}^{\mathrm{T}} E\left[\dot{q}_{2k}^2\left(\frac{\partial V}{\partial H_{2k}}\right)^2\right]\right)P_2\Bigg] R_{pi}^{-\mathrm{T}},
\end{aligned}
\tag{34d}
$$

where $\phi_{jk}^i$ is the $i$th element of mode vector $\phi_{jk}$ of structure $j$, $m_{jii}$ is the $i$th diagonal element of mass matrix $M_j$, $R_{pi}^{-1}$ denotes the $i$th row vector of inverse matrix $R_p^{-1}$ and

$$
E\left[\dot{q}_{jk}^2\left(\frac{\partial V}{\partial H_{jk}}\right)^2\right] = \int_0^{+\infty} H_{jk}\left(\frac{\partial V}{\partial H_{jk}}\right)^2 p(H_1, H_2)\,\mathrm{d}H_1\,\mathrm{d}H_2. \tag{35}
$$

The response statistics of the corresponding uncontrolled structures under random seismic excitation can be obtained in the same way by making the control forces vanishing. At last, the following performance criteria [4–6] are used to evaluate the control efficacy:

$$
K = \frac{RMS(\text{Response}_u) - RMS(\text{Response}_c)}{RMS(\text{Response}_u)}, \tag{36a}
$$

$$
\mu = \frac{K}{\sum_{i=1}^{n_3} RMS(u_i^*)/\left[\sigma\left(\sum_{i=1}^{n_1} m_{1ii}+\sum_{i=1}^{n_2} m_{2ii}\right)\right]}, \tag{36b}
$$

where $RMS(\cdot)$ denotes the root-mean-square operator and the subscripts $u$ and $c$ denote the uncontrolled and controlled values, respectively. The ratio $K$ measures the relative response reduction of the controlled and uncontrolled structural systems or the control effectiveness. The ratio $\mu$ measures the relative response reduction per normalized control force or the control efficiency. The higher $K$ and $\mu$ indicate the control method with more response mitigation capabilities.

## 5. Numerical results

A numerical study is conducted on the non-linear stochastic optimal control of coupled adjacent building structures consisting of a 20-storey building and a 10-storey building with a few control devices. The mass of each floor is $1.6 \times 10^6$ kg, the interstorey stiffness is $1.2 \times 10^{10}$ N/m and the modal damping ratio is 0.02. The first six natural frequencies of the 20-storey building structure are 1.06, 3.16, 5.25, 7.30, 9.32 and 11.28 Hz, and the first four natural frequencies of the 10-storey building structure are 2.06, 6.13, 10.07 and 13.78 Hz. The spectral parameters of random seismic excitation are taken as $\sigma^2 = 0.6\ \text{m}^2/\text{s}^3$, $\omega_g = 19$ rad/s and $\zeta_g = 0.2$ unless otherwise mentioned. The numbers of structural modes used for response analysis $m_1 = 6$ and $m_2 = 4$ while the controlled mode numbers $m_3 = 3$ and $m_4 = 2$. The weight coefficients of control forces and modal energies are $R = \text{diag}\{10, 10, 8, 3, 2\}$, $S^a_{1i} = 0$, $S^a_{2i} = 0$, $[S^c_{1i}] = [0.08, 0.1, 0.04]$, $[S^c_{2i}] = [0.08, 0.04]$, $S^b_{1ij} = 0$, $S^b_{2ij} = 0$ and $S^b_{3ij} = 0$. Some numerical results are displayed in Figs. 2–6 and in Tables 1–4.

Fig. 2 shows the performance criteria $K$ and $\mu$ of displacements and interstorey drifts of the coupled structures by using the proposed control method when the control device connects the two adjacent buildings only at the 10th floor level. About 60% displacement response reduction ($K$) with 0.85 efficiency ($\mu$) at the middle of taller building and 55% response reduction ($K$) with 0.80 efficiency ($\mu$) for shorter building are achieved. The response pattern of interstorey drifts is similar to that of corresponding displacements and then only the numerical results of interstorey drifts are given in the following.

The effect of seismic excitation features on the control efficacy is studied with the control device at the 10th floor level. Fig. 3 illustrates the relative response reduction $K$ and control efficiency $\mu$

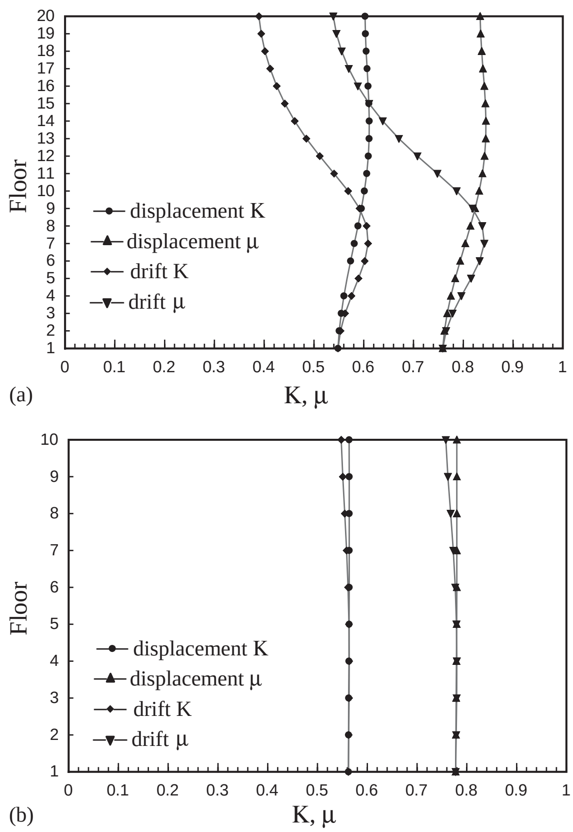
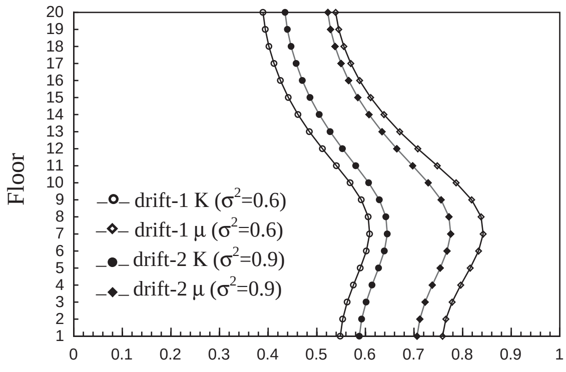



Fig. 2. Relative reduction $K$ and efficiency $\mu$ of displacements and interstorey drifts. (a) Taller building; (b) shorter building.



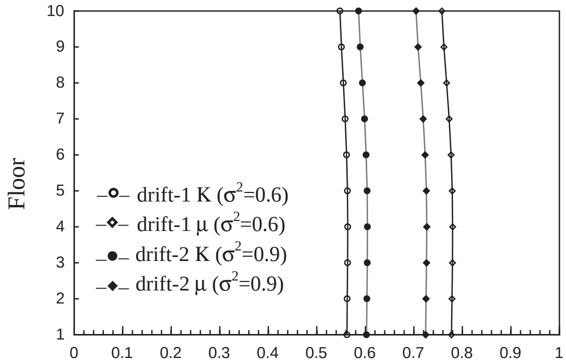




Fig. 3. Relative reduction $K$ and efficiency $\mu$ of interstorey drifts under different $\sigma^2$ $(m^2/s^3)$. (a) Taller building; (b) shorter building.

of interstorey drifts under different excitation intensity $\sigma$. With the increase of intensity $\sigma$, the response reduction capability is enhanced while the efficiency is decreased. Fig. 4 shows the relative response reduction and control efficiency of interstorey drifts under different dominant excitation frequency $\omega_g$. The response reduction or mitigation capability increases as the dominant frequency $\omega_g$ is close to the structural natural frequency [for the 20-storey building structure, $\omega_g = 1.27$ Hz and the natural frequency = 1.06 Hz, see Fig. 4(a); for the 10-storey building structure, $\omega_g = 2.23$ Hz and the natural frequency = 2.06 Hz, see Fig. 4(c)], even though the efficiency has a little decrease.

The effect of control device placement and number on the control efficacy is also studied. Fig. 5 illustrates the relative response reduction and control efficiency of interstorey drifts when a single control device is placed at the 10th floor, the 8th floor or the 6th floor. The response mitigation capability of the control device at the 10th floor level as well as the efficiency is better than at the others. This result means the optimum position of control devices close to the floor level of the largest amplitude of dominant structural modes. Fig. 6 shows the relative response reduction and control efficiency of interstorey drifts for different control device number and placement (three

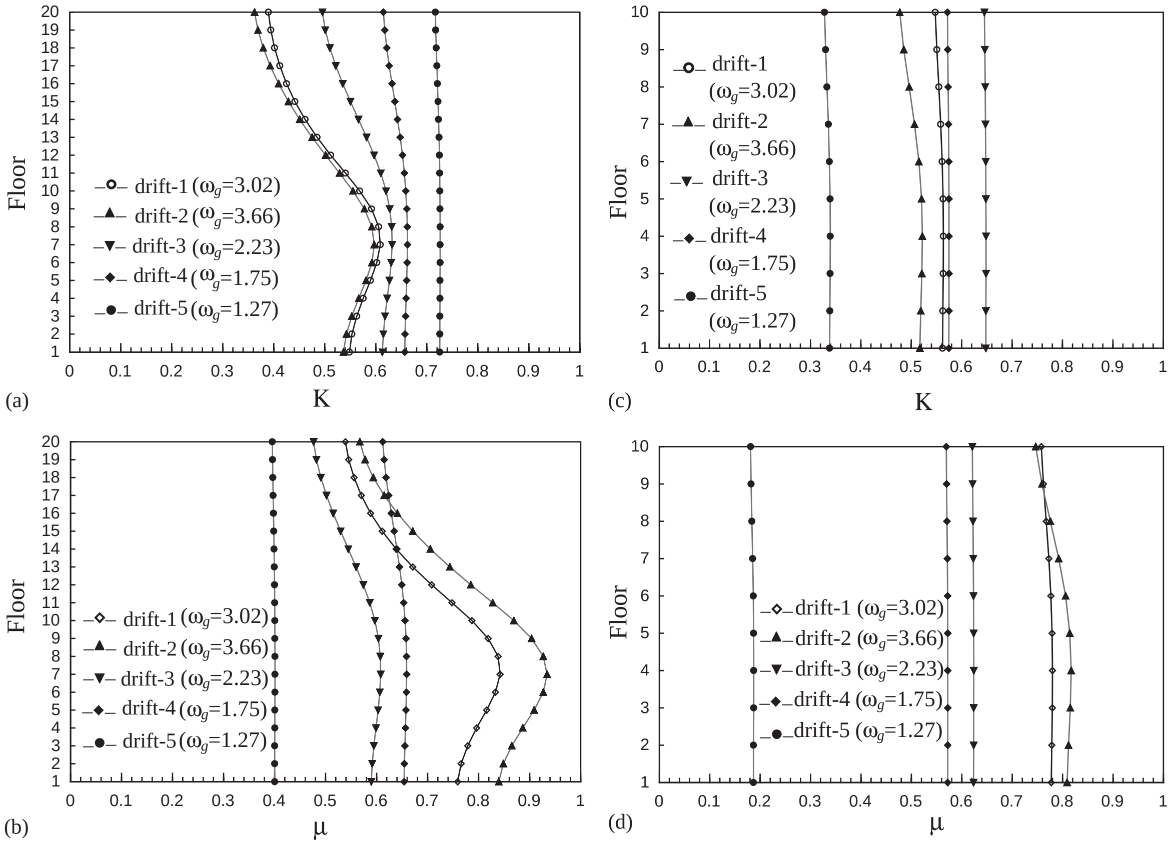



Fig. 4. Relative reduction $K$ and efficiency $\mu$ of interstorey drifts under different $\omega_g$ (Hz). (a) Taller building: $K$; (b) taller building: $\mu$; (c) shorter building: $K$; (d) shorter building: $\mu$.

cases: one control device at the 10th floor; two control devices at the 10th and 8th floors; three control devices at the 10th, 8th and 6th floors, respectively). The response reduction capability does not increase with using more control devices at lower floors due to the interaction among control devices. A similar observation is made for base shears as given in Tables 1–4.

The linear optimal control of coupled adjacent structures under seismic excitation has been studied [14] and the linear control force consists of viscous damping force component and linear restoring force component, which can be determined by optimizing damping and stiffness. It has been obtained that the non-linear optimal control method is more effective than linear one for structural response reduction [2–6]. The random seismic response control of coupled adjacent building structures using non-linear hysteretic dampers has been researched by optimum analysis [12] and the maximum relative reduction of root-mean-square interstorey drifts would be less than 30%. However, about 55% relative reduction of root-mean-square interstorey drifts (see Fig. 2) can be achieved by using the proposed non-linear stochastic optimal control method.

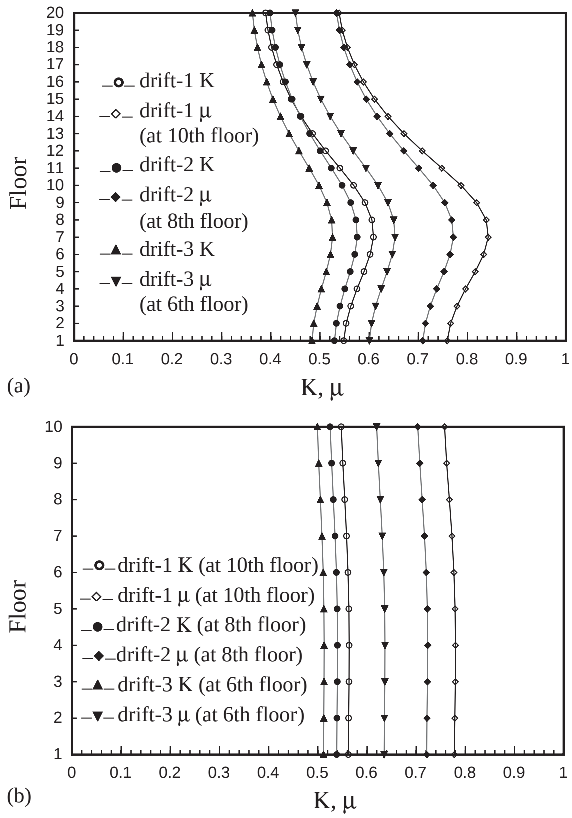



Fig. 5. Relative reduction $K$ and efficiency $\mu$ of interstorey drifts for various control device positions. (a) Taller building; (b) shorter building.

## 6. Conclusions

Coupled structures under random excitation are modelled as a quasi-integrable Hamiltonian system of multi-degree-of-freedom. A non-linear stochastic optimal control method for the system has been developed based on the stochastic dynamical programming principle and the stochastic averaging method. The random seismic response control of coupled adjacent building structures has been studied by using the developed method. The method has the following advantages: (a) the system energy control generally instead of the system state control is conducted based on the stochastic averaging method of energy envelope and then the dimension of the control problem is reduced; (b) the effect of random excitation spectra on the optimal control law is taken into account according to the stochastic dynamical programming principle and the derived dynamical programming equation has a classical solution of value function; (c) the optimal control forces can be obtained in the form of generalized non-linear damping forces as given in the coupled structural system control, which can be produced actually by active dampers; (d) it is applicable to many engineering structures as quasi-integrable Hamiltonian systems of multi-degree-of-freedom as illustrated by the random seismic response control of the coupled structures.

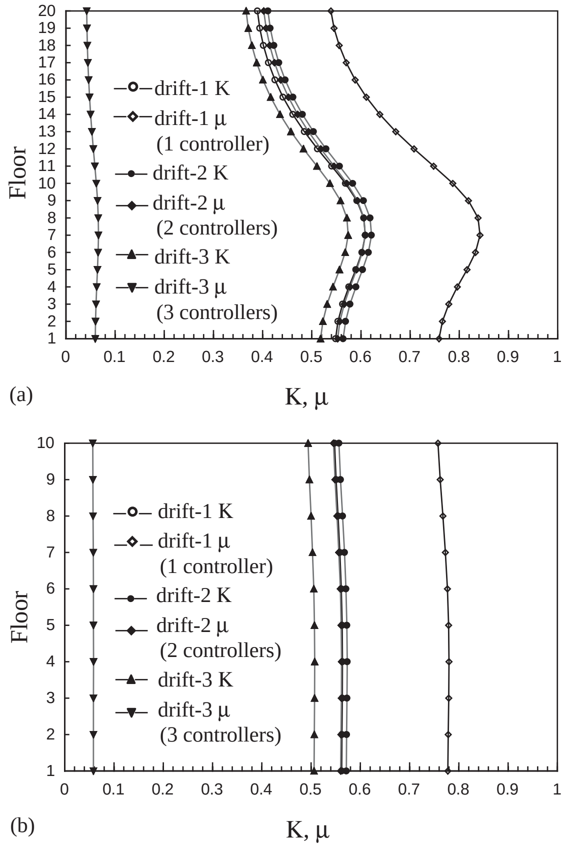



Fig. 6. Relative reduction $K$ and efficiency $\mu$ of interstorey drifts for various control device numbers. (a) Taller building; (b) shorter building.

Table 1
Relative reduction $K$ and efficiency $\mu$ of base shears under different $\sigma$

| $\sigma^2$ (m$^2$/s$^3$) | Taller building | | Shorter building | |
|---|---|---|---|---|
| | $K$ | $\mu$ | $K$ | $\mu$ |
| 0.6 | 0.796 | 1.10 | 0.808 | 1.12 |
| 0.9 | 0.830 | 1.00 | 0.842 | 1.01 |

Numerical results for the coupled structures with the proposed control method show that the more random response reduction can be achieved by using a few control devices and the response reduction capability can increase with random excitation intensity. In consequence, the proposed non-linear stochastic optimal control method is potentially promising for structural control applications.

Table 2
Relative reduction $K$ and efficiency $\mu$ of base shears under different $\omega_g$

| $\omega_g$ (Hz) | Taller building | | Shorter building | |
|---|---|---|---|---|
| | $K$ | $\mu$ | $K$ | $\mu$ |
| 1.27 | 0.925 | 0.51 | 0.562 | 0.31 |
| 1.75 | 0.882 | 0.88 | 0.819 | 0.82 |
| 2.23 | 0.850 | 0.82 | 0.876 | 0.84 |
| 3.02 | 0.796 | 1.10 | 0.808 | 1.12 |
| 3.66 | 0.785 | 1.23 | 0.767 | 1.20 |

Table 3
Relative reduction $K$ and efficiency $\mu$ of base shears for various control device positions

| Position of control device | Taller building | | Shorter building | |
|---|---|---|---|---|
| | $K$ | $\mu$ | $K$ | $\mu$ |
| 10th floor | 0.796 | 1.10 | 0.808 | 1.12 |
| 8th floor | 0.779 | 1.04 | 0.787 | 1.05 |
| 6th floor | 0.734 | 0.91 | 0.761 | 0.95 |

Table 4
Relative reduction $K$ and efficiency $\mu$ of base shears for various control device numbers

| Control devices | | Taller building | | Shorter building | |
|---|---|---|---|---|---|
| Number | Position | $K$ | $\mu$ | $K$ | $\mu$ |
| 1 | 10th floor | 0.796 | 1.10 | 0.808 | 1.12 |
| 2 | 10th & 8th floors | 0.810 | 0.79 | 0.816 | 0.80 |
| 3 | 10th, 8th & 6th floors | 0.768 | 0.09 | 0.756 | 0.09 |

## Acknowledgements

This study was supported by the National Natural Science Foundation of China under grant No. 19972059, the Zhejiang Provincial Natural Science Foundation of China under grant No. 101046 and the Hong Kong Polytechnic University through the Area of Strategic Development Programme. The support is gratefully acknowledged.

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